

Exact Wavefunctions for a Delta Function Bose Gas with Higher Derivatives

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Abstract

A quantum mechanical system describing bosons in one space dimension with a kinetic energy of arbitrary order in derivatives and a delta function interaction is studied. Exact wavefunctions for an arbitrary number of particles and order of derivative are constructed. Also, equations determining the spectrum of eigenvalues are found.

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Physical systems whose equations contains derivatives higher than two arise in a number of contexts. In classical mechanics, for example, the Korteweg-deVries equation, which describes shallow water waves, contains a third order derivative term [1]. Higher derivative theories of classical gravity, first introduced by Weyl [2], have some attractive cosmological properties, such as gravity driven expansion [3]. A model of quantum gravity with higher derivative terms has been shown to be renormalizable [4] and asymptotically free [5]. In particle physics, the Nambu-Jona-Lasinio [6], Schwinger [7], and Skyrme [8] models with higher derivative terms added have been considered. Also, in the superfield formulation of supersymmetric field theories, higher derivative terms arise naturally [9].

In this work, a family of higher derivative generalizations of Schrodinger's equation is studied. The system describes bosons where the "kinetic energy" is a polynomial of order n in derivatives

and the interaction is a delta function potential. A particular choice of polynomial corresponds to the much studied delta function bose gas (or quantum nonlinear Schrodinger equation)[10, 11]. The main result here is that for an arbitrary polynomial in derivatives and arbitrary number of particles, exact eigenfunctions for the system may be constructed.

Formulating the system under study, consider the eigenvalue problem

$$H\Psi = E\Psi, \quad (1)$$

where the hamiltonian H is the partial differential operator

$$H = \sum_{j=1}^N \sum_{l=1}^n a_l (-i\partial_{x_j})^l + \sum_{j < k} 2c\delta(x_j - x_k), \quad (2)$$

and $\Psi(x_1, \dots, x_N)$ is a symmetric function of the x_j . N is the number of bosons, n is an arbitrary natural number, the a_l are arbitrary real coefficients, and c is a real interaction constant. Equation (1) is a generalization of Schrodinger's equation describing bosons interacting via a delta function where the usual 2nd order kinetic energy has been replaced with a polynomial containing higher order derivatives.

The hamiltonian (2) is translationally invariant, thus the momentum operator

$$P = -i \sum_{j=1}^N \partial_{x_j} \quad (3)$$

commutes with the hamiltonian (2).

If $a_2 = 1$, and all other $a_l = 0$, then (2) is called the δ -function bose gas, and the eigenfunctions for arbitrary N were first found by Lieb and Liniger [10]. The method of solution has come to be known as coordinate Bethe ansatz, after Hans Bethe, who first used the idea to find the eigenstates of the Heisenberg spin chain [12], and has since been applied to many 1+1 dimensional quantum and 2 dimensional statistical mechanical systems [13].

Proceeding now with constructing the the eigenfunctions of (1), first consider a simpler model

where $a_n = 1$ and all other a_l are equal to zero

$$H = \sum_{j=1}^N (-i\partial_{x_j})^n + \sum_{j < k} 2c\delta(x_j - x_k) = T_n + V. \quad (4)$$

The formulae for the eigenfunctions of this model are simpler and it will be seen that the solution for the general model (2) can be easily constructed knowing the solution of (4).

Next, note that for coordinate values where no two x_j 's coincide, $x_j \neq x_k$, the interaction term V is zero and the eigenvalue equation (1) reduces to

$$\sum_{j=1}^N (-i\partial_{x_j})^n \Psi = E\Psi. \quad (5)$$

A “basic” solution of this equation is

$$e(x_1, \dots, x_N; k_1, \dots, k_N) = \exp i\left(\sum_{j=1}^N k_j x_j\right), \quad (6)$$

where the k_j are a set of quasimomentum parameters. The energy and momentum of this state are

$$E = \sum_{j=1}^N k_j^n, \quad P = \sum_{j=1}^N k_j. \quad (7)$$

The effects of the interaction term V may be rewritten as a set of conditions which Ψ must satisfy when crossing a boundary $x_j = x_k$. To obtain these conditions, integrate equation (1) across a boundary $x_j = x_k$:

$$\lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} d(x_k - x_j) H \Psi = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} d(x_k - x_j) E \Psi \quad (8)$$

The RHS is zero, leaving only the kinetic and interaction terms. Calculating the integral

$$\lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} d(x_k - x_j) (T_n + V) \Psi, \quad (9)$$

yields the equation

$$\frac{(-i)^n}{2^{n-1}} \sum_{l=0}^n \binom{n}{l} (1 + (-1)^{n-l}) ((\partial_{x_k} + \partial_{x_j})^l (\partial_{x_k} - \partial_{x_j})^{n-l-1} \Psi)|_{x_k=x_j} + c\Psi(x_k = x_j) = 0, \quad (10)$$

where $\binom{n}{l}$ is the usual binomial coefficient. Thus, the original problem of solving (1) has been recast as that of solving the noninteracting equation (5) subject to the conditions (10).

To simplify the task of solving (5) and (10), note that since this system describes bosons on a line, it is sufficient to construct the eigenfunction only in an ordered sector,

$$x_{Q_1} < x_{Q_2} < \dots < x_{Q_{N-1}} < x_{Q_N}, \quad (11)$$

where Q is a permutation of N elements. The value of the wavefunction for other coordinate orderings can be obtained by symmetric continuation. For convenience, we set $Q = 1$.

Next, consider the following superposition of the solutions (6)

$$\Psi = \sum_{P \in S_N} \phi(k_1, \dots, k_N; P) \exp i \left(\sum_{j=1}^N k_{P_j} x_j \right), \quad (12)$$

where k_1, \dots, k_N are quasimomenta, P is a permutation of N elements, and $\phi(k_1, \dots, k_N; P)$ are $N!$ undetermined phase functions of the quasimomenta and P . The energy and momentum of this state are given by (7).

Now it will be determined how to choose the $\phi(P)$ so that the boundary condition (10) is satisfied across an intersection $x_j = x_{j+1}$. Consider the partial sum

$$\tilde{\Psi} = \phi(P) \exp i \left(\sum_{j=1}^N k_{P_j} x_j \right) + \phi(P') \exp i \left(\sum_{j=1}^N k_{P'_j} x_j \right), \quad (13)$$

where P' differs from P by the exchange of k_{P_j} and $k_{P_{j+1}}$. If this partial sum satisfies the boundary conditions (10), then so does the the entire sum (12) since given a pair $j, j+1$, then $N!$ permutations may be split into $\frac{N!}{2}$ pairs of the type in expression (13). Now requiring that the partial sum (13) satisfies (10), one obtains a set of conditions on the phases $\phi(P)$

$$\frac{-i}{2^{n-1}} \sum_{l=0}^n \binom{n}{l} (1 + (-1)^{n-l}) (k_{P_j} + k_{P_{j+1}})^l (k_{P_{j+1}} - k_{P_j})^{n-l-1} (\phi(P) - \phi(P')) + c(\phi(P) + \phi(P')) = 0. \quad (14)$$

To write this in a more transparent form, define the polynomial

$$P_n(k, k') = \frac{1}{2^{n-1}} \sum_{l=0}^n \binom{n}{l} (1 + (-1)^{n-l}) (k' + k)^l (k' - k)^{n-l-1}. \quad (15)$$

Note that this polynomial is of degree $n - 1$ and odd under the interchange k and k' . Rewriting (14) in terms of the ratio of phases, one obtains

$$\frac{\phi(P')}{\phi(P)} = \frac{P_n(k_{P_j}, k_{P_{j+1}}) + ic}{P_n(k_{P_j}, k_{P_{j+1}}) - ic} \equiv \theta_n(k_{P_j}, k_{P_{j+1}}). \quad (16)$$

These equations have the interpretation as the phase accrued when two particles scatter off one another. Note that the scattering phase has the property $1/\theta_n(k, k') = \theta_n(k', k)$.

Lemma: A solution to the conditions (16) is

$$\phi(P) = \prod_{l>m, P_l < P_m} \theta_n(k_{P_l}, k_{P_m}). \quad (17)$$

Thus, the result for the eigenfunctions of (4) is

$$\Psi = \sum_{P \in S_N} \phi(k_1, \dots, k_N; P) \exp i \left(\sum_{j=1}^N k_{P_j} x_{Q_j} \right), \quad (18)$$

with $\phi(P)$ chosen according to (17), locally satisfies the eigenvalue equation (1) for all natural n and N . The eigenvalues of H and P are given by (7).

Properties of the solution:

1. The wavefunction Ψ is continuous for all x_j , and the first derivative $\partial_{x_j} \Psi(x)$ is discontinuous when any two x 's coincide
2. $\Psi(k_j = k_l) = 0$ for all j, k , thus a Pauli-type exclusion principle holds for the quasimomenta parameters.
3. Explicit formulae for the first few θ_n are

$$n = 1 : \quad \theta_1 = -1 \quad (19)$$

$$n = 2 : \quad \theta_2 = \frac{k - k' + ic}{k - k' - ic} \quad (20)$$

$$n = 3 : \quad \theta_3 = \frac{\frac{3}{2}(k^2 - k'^2) + ic}{\frac{3}{2}(k^2 - k'^2) - ic} \quad (21)$$

$$n = 4 : \quad \theta_4 = \frac{\frac{1}{4}(7k^3 + 3k^2k' - 3kk'^2 - 7k'^3) + ic}{\frac{1}{4}(7k^3 + 3k^2k' - 3kk'^2 - 7k'^3) - ic} \quad (22)$$

$$n = 5 : \quad \theta_5 = \frac{\frac{1}{8}(15k^4 + 10k^3k' - 10kk'^3 - 15k'^4) + ic}{\frac{1}{8}(15k^4 + 10k^3k' - 10kk'^3 - 15k'^4) - ic} \quad (23)$$

The case $n = 1$ corresponds to free fermions with a linear energy spectrum.

Next, to construct the solution of the general model (2), with arbitrary a_l , note that since the differential equation (1) is linear, all polynomials corresponding to different values of l simply add. So, stating the main result:

Theorem: the eigenfunctions of (2) are of the form

$$\Psi = \sum_{P \in S_N} \Phi(k_1, \dots, k_N; P) \exp i \left(\sum_{j=1}^N k_{P_j} x_{Q_j} \right), \quad (24)$$

where

$$\Phi(P) = \prod_{l > m, P_l < P_m} \Theta_n(k_{P_l}, k_{P_m}), \quad (25)$$

and the two body scattering phase is

$$\Theta_n(k, k') = \frac{\sum_{l=1}^n a_l P_l(k, k') + ic}{\sum_{l=1}^n a_l P_l(k, k') - ic}. \quad (26)$$

Θ_n obeys the properties $1/\Theta(k, k') = \Theta(k', k)$ and $\Theta(k, k) = -1$. The latter implies that the Pauli exclusion principle for spectral parameters holds for the general model. The eigenvalues of H and P are given by

$$E = \sum_{j=1}^N \sum_{l=1}^n a_l (k_j)^l, \quad P = \sum_{j=1}^N k_j. \quad (27)$$

Thus far, the boundary conditions on the partial differential equation (1) acts have been left unspecified. The next step is to impose that the wavefunction Ψ is periodic in a box of length L , ie., require that

$$\Psi(x_j + L) = \Psi(x_j) \quad j = 1, \dots, N. \quad (28)$$

This condition imposes the constraint that the parameters k_1, \dots, k_N must satisfy a set of coupled equations

$$e^{ik_j L} = - \prod_{l=1}^N \Theta_n(k_j, k_l) \quad j = 1, \dots, N. \quad (29)$$

An interpretation of these equations is as the phase acquired by particle j as it traverses around the box back to its starting point, scattering off the other particles along the way.

To understand the ground state, excitation spectrum, and other physical characteristics of the hamiltonian (4), one needs to analyze the solutions of the equations (29). The properties of equations (29) can differ essentially from those of equations found for other $1 + 1$ dimensional quantum systems (e.g. those of [13]). In particular, for complex solutions, the imaginary part of a solution can have a nontrivial functional dependence on the real part. The properties of these complex solutions need to be understood in order to study physical properties which depend on the spectrum of eigenvalues.

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